Computing Periods...

Junhee Cho and Martin Ziegler, KAIST

Abstract. A *period* is the difference between the volumes of two semi-algebraic sets. Recent research has located these in low levels of the Grzegorczyk Hierarchy, that is, established a structural complexity-theoretic upper bound. The present extended abstract describes work in progress on their refined resource-oriented parameterized computational complexity.

An open question in Algebraic Model Theory asks for a real number which is not a *period*, that is, not 'definable' as the difference between two volumes of semi-algebraic sets; see [KZ01, Problem 3]. Recall that semi-algebraic means (a finite Boolean combination of) sets of solutions polynomial inequalities with integer coefficients

$$S_p \;:=\; \big\{(x_1,\ldots,x_d)\;\big|\;p(x_1,\ldots,x_d)>0\big\}\;\subseteq\;\mathbb{R}^d, \quad p\in\mathbb{Z}[x_1,\ldots,x_d]\;. \tag{1}$$

The family of periods includes transcendental $\pi=\mathrm{vol}\{(x,y):x^2+y^2<1\}$ yet is countable, hence missing most reals. Towards an explicit example, a recent series of works has gradually narrowed down periods to be computable in the sense of Recursive Analysis, of *elementary* computational complexity [Yos08] in the Grzegorczyk Hierarchy, and in fact even lower:

Fact 1 *Let* Lower Elementary *be the smallest class of total multivariate functions $f:\mathbb{N}^d\to\mathbb{N}=\{0,1,2,\ldots\}$ containing the constants, projections, successor, modified difference $x\dot{-}y=\max\{x-y,0\}$, and is closed under composition and bounded summation $f(\vec{x},y)=\sum_{z=0}^{y}g(\vec{x},z)$.*

Write $\mathcal{M}^2=\mathcal{E}^2$ for the smallest class of such f containing the constants, projections, successor, modified difference, binary multiplication, and is closed under composition and bounded search $\mu(f)(\vec{x},y)=\min\{z\le y: f(\bar{z},z)=0\}$.

A real number r is lower elementary*/in $\mathcal{M}^2$ if there exist lower elementary/$\mathcal{M}^2$ functions $f,g,h:\mathbb{N}\to\mathbb{N}$ with $\left|r-\frac{f(N)-g(N)}{h(N)}\right|<1/N$ for all $N>0$.*

- *a) All functions from $\mathcal{M}^2$ are lower elementary; and the latter functions grow at most polynomially in the value of the arguments. In terms of the binary input length and with respect to bit-cost, lower elementary functions are computable using a linear amount of memory for intermediate calculations and output, that is, they belong to the complexity class $\mathsf{FSPACE}(n)$.*
- *b) $\mathsf{FSPACE}(n)$ is closed under bounded summation and therefore coincides with the class of lower elementary functions. The 0/1-valued functions (that is, decision problems) in $\mathcal{M}^2$ exhaust the class $\mathsf{SPACE}(n)$* [Rit63, §4]*; cmp.* [Kut87]*.*
- *c) π and $e=\sum_n 1/n!$ and Liouville's transcendental number $L=\sum_n 10^{-n!}$ and the Euler-Mascheroni Constant $\gamma=\lim_n\big(-\ln(n)+\sum_{k=1}^{n}1/k\big)$ are all lower elementary* [Sko08, §3]*.*
- *d) The set of lower elementary real numbers constitutes a real closed field: Binary sum and product and reciprocal of lower elementary real numbers, as well as any real root of a non-zero polynomial with lower elementary coefficients, are again lower elementary* [SWG12, Theorem 2]*.*
- *e) Arctan, natural logarithm and exponential as well as Γ and ζ function map lower elementary reals to lower elementary reals* [TZ10, §9]*.*
- *f) Natural logarithm maps periods to periods; $\zeta(s)$ is a period for every integer $s\ge 2$* [KZ01, §1.1]*.*
- *g) Periods are lower elementary* [TZ10, Corollary 6.4]*.*
- *h) Given a Boolean expression $\varphi(x_1,\ldots,x_m)$ as well as the degrees and coefficients of the polynomials p_j defining its constituents S_{p_j}, deciding whether the semi-algebraic set $\varphi\big(S_{p_1},\ldots,S_{p_m}\big)$ is non-empty/of given dimension* [Koi99] *is complete for the complexity class $\mathsf{NP}^0_{\mathbb{R}}\supseteq\mathsf{NP}$.*

Item a) follows by structural induction. Together with b) it relates resource-oriented to Grzegorczyk's structural Complexity Theory. Common efficient and practical algorithms tailored for approximating L, e, γ, or the period π do so up to absolute error $1/N := 2^{-n}$ within time polynomial in the binary precision parameter $n = \log_2 N$ [Kan03]; whereas the best runtime bound known for $\mathsf{SPACE}(n)$ is only exponential [Pap94, Problem 7.4.7]. Note that the hardness Result h) does not seem to entail a lower bound on the problem of approximating the volume.

This raises the question, driving the present work in progress, of whether or not periods in general admit polynomial-time algorithms; and how/what further parameters affect their computational bit-complexity in addition to the binary output precision n [Ko91, Wei03]. Indeed we agree [KZ01, Problem 2] that efficient Reliable High-Precision Numerics and Experimental *Transcendental* Mathematics as computational tools can provide enriched insight into questions including, but not restricted to [Ret12], explicit candidates for non-periods.

We restrict to (volumes of) semi-algebraic sets inside the unit cube $[0;1)^d$. One approach to the 1D case $d = 1$ subdivides the interval $[0;1)$ into sub-intervals $[a \cdot 2^{-n}, (a+1)\cdot 2^{-n})$, $\mathbb{N} \ni a < 2^n$; evaluates the polynomial(s) signs on a random point from each sub-interval; and counts those with positive sign, divided by 2^n: Since a polynomial of degree k can have at most k roots, this will approximate the true volume up to error $k \cdot 2^{-n}$. Moreover with high probability a random point will avoid all roots, hence rendering the sign computable; cmp. [MPPZ16, Definition 2]. The following suggests a way of generalizing this to higher dimensions:

Lemma 2. *Fix a d-variate real or complex power series around zero* $f(\vec{x}) = \sum_{\vec{j}} c_{\vec{j}} \cdot x_1^{j_1} \cdots x_d^{j_d}$ *with $\vec{j} = (j_1, \ldots, j_d)$ ranging over $\mathbb{N}^d$, converging absolutely and uniformly for all $\vec{x} = (x_1, \ldots, x_d) \in [-R, +R]^d$. Abbreviate $|\vec{x}| := |x_1| + \cdots + |x_d|$ and for $0 < r < R$ consider the condition*

$$|c_{\vec{0}}| \quad > \quad \sum_{\vec{j} \neq \vec{0}} |c_{\vec{j}}| \cdot r^{|\vec{j}|} \quad . \tag{2}$$

a) If $0 < r < R$ satisfies Condition (2), then f has no root in $[-r, +r]^d$.
b) If $f(\vec{0}) \neq 0$, then there exists $r > 0$ such that Condition (2) holds.
c) Suppose f is a polynomial of total degree $k = \max\{j_1 + \cdots + j_d : c_{j_1,\ldots,j_d} \neq 0\}$ and consider the N^d cubes $\prod_{j=1}^{d} [A_j/N, (A_j+1)/N) \subseteq [0,1)$ in $[0,1)^d$, $A_1, \ldots, A_d \in \{0, 1, \ldots, N-1\} =: [N]$. Then at most $\mathcal{O}\big(k + d^2 \cdot N\big)^{d-1} \cdot (k + d^2)$ of them contain a root of f.

References

[Kan03] Yasumasa Kanada. 計算機による円周率計算（特集 円周率 π）. *J. Mathematical Culture*, 1(1):72–83, 2003.

[Ko91] Ker-I Ko. *Complexity Theory of Real Functions.* Progress in Theoretical Computer Science. Birkhäuser, Boston, 1991.

[Koi99] Pascal Koiran. The real dimension problem is $\mathrm{NP}_{\mathbb{R}}$-complete. *Journal of Complexity*, 15(2):227–238, 1999.

[Kut87] Miroslaw Kutylowski. Small Grzegorczyk classes. *Journal of the London Mathematical Society*, 36(2):193–210, 1987.

[KZ01] Maxim Kontsevich and Don Zagier. Periods. In Björn Engquist and Wilfried Schmid, editors, *Mathematics unlimited — 2001 and beyond*, pages 771–808. Springer, 2001.

[MPPZ16] Norbert Müller, Sewon Park, Norbert Preining, and Martin Ziegler. On formal verification in imperative multivalued programming over continuous data types. *arXiv*, 1608.05787, 2016.

[Pap94] Christos H. Papadimitriou. *Computational Complexity.* Addison-Wesley, 1994.

[Ret12] Robert Rettinger. On computable approximations of Landau's constant. *Log. Methods Comput. Sci.*, 8(4):4:15, 11, 2012.

[Rit63] Robert W. Ritchie. Classes of predictably computable functions. *Trans. Amer. Math. Soc.*, 106(1):139–173, 1963.

[Sko08] Dimiter Skordev. On the subrecursive computability of several famous constants. *Journal of Universal Computer Science*, 14(6):861–875, 2008.

[SWG12] Dimiter Skordev, Andreas Weiermann, and Ivan Georgiev. $\mathcal{M}^2$-computable real numbers. *J. Logic Comput.*, 22(4):899–925, 2012.

[TZ10] Katrin Tent and Martin Ziegler. Computable functions of reals. *Münster J. Math.*, 3:43–65, 2010.

[Wei03] Klaus Weihrauch. Computational complexity on computable metric spaces. *Mathematical Logic Quarterly*, 49(1):3–21, 2003.

[Yos08] Masahiko Yoshinaka. Periods and elementary real numbers. *arXiv*, 0805.0349, 2008.